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A disorder solution for a generalised mixed-spin model on anisotropic Kagome lattices

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Abstract. A disorder solution is given for a generalised mixed-spin model on an anisotropic Kagome lattice. Because of its lower rotational symmetry, the solution is obtained by decimation along four different directions. In the case of a spin-$\frac{1}{2}$ and spin-1 mixed model, the intra-row correlations in both horizontal and vertical directions are zero when a relation imposed on the interactions is satisfied.

1. Introduction

Very recently a number of papers have appeared that deal with disorder solutions of many different kinds of models (Rujan 1984, Baxter 1984, Jaekel and Maillard 1985, Wu 1985, Giacomini 1986). These models are known to possess remarkable submanifolds in the space of parameters, where the partition function is computable and takes a very simple algebraic form (Enting 1977). Disorder solutions provide an important insight into the analytical behaviour of the partition function in its anisotropic parameters. Some information in the vicinity of the disorder solution can be obtained through a new perturbative expansion (Georges *et al* 1986). Moreover, Georges and Le Doussal (1987) have recently shown that $(D+1)$-dimensional equilibrium spin models satisfying a 'disorder condition' may be equivalent to D-dimensional probabilistic cellular automata (PCA), which allows us to provide techniques from the equilibrium statistical mechanics of spin models to the study of statistical properties of the dynamical behaviour of PCA.

It is well known that symmetry plays an important role in statistical mechanics, especially in critical phenomena. Recently, Tang and Hu (1987a, b) have considered a generalised mixed-spin (GMS) model which has lower translational symmetry. Its phase diagram is described via renormalisation group approximations (Tang and Hu 1987a). The known exact results include a disorder solution on a checkerboard square lattice (Tang and Hu 1987b) and an Ising transition critical line on honeycomb lattices in a parameter subspace (Tang and Hu 1988).

In this paper, a Kagome lattice GMS model, which has the same translational symmetry as, but lower rotational symmetry than, the pure spin system, is considered. Much interest is given to this model in our research because of its plentiful properties of symmetry, even in the case of the spin-$\frac{1}{2}$ Ising model which has been considered by Jaekel and Maillard (1985). In their work, a disorder solution for this spin-$\frac{1}{2}$ model

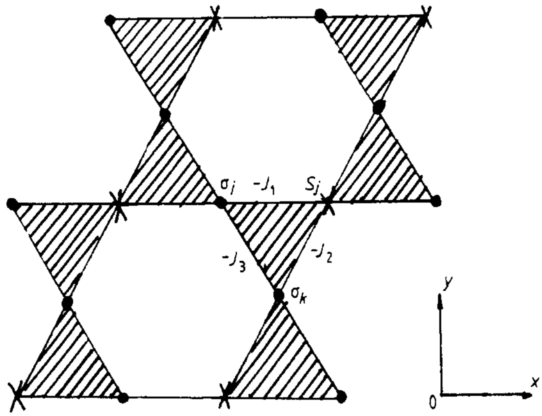

Figure 1. The anisotropic Kagome lattice GMS model. The GMS model consists of spin-$\frac{1}{2}$ and spin-1 Ising objects, which are indicated by circles (●) and crosses (×), respectively. Each shaded cell is bordered by two sets of interactions, $-J_1$, $-J_2$ and $-J_3$.

is obtained by decimation along one direction. Because of the low rotational symmetry of the Kagome lattice, a different decimation direction (the direction along which the decimation is performed) may lead to a different disorder solution, which is also true even in the case of a pure spin-$\frac{1}{2}$ Ising model. Therefore, the disorder solution for this model should include all the solutions obtained in different decimation directions. Along this line, the exact decimation method, as used by Jaekel and Maillard (1985) and Wu (1985), is applied to the anisotropic Kagome lattice GMS model.

The outline of the paper is as follows. The anisotropic Kagome lattice GMS model is introduced in § 2. Section 3 is devoted to results and a discussion is given in § 4.

2. The model

In order to examine the influence of symmetry on disorder solutions, we consider a generalised mixed-spin (GMS) model on anisotropic Kagome lattices with the Hamiltonian

$$\mathscr{H}\{\sigma, S\} = -J_1 \sum_{i,j} \sigma_i S_j - J_2 \sum_{k,j} \sigma_k S_j - J_3 \sum_{i,k} \sigma_i \sigma_k - G \sum_i S_i^2 \tag{1}$$

where $\sigma = \pm\frac{1}{2}$ and $S = 0, \pm 1$. The first three summations are taken over three different directions, respectively. This Hamiltonian reduces to the Ising model as $G \to \infty$, the mixed-spin model as $G \to 0$ and a site-diluted Ising model as $G < 0$. Even for the existence of spin-1, the translational symmetry is sustained, whereas the rotational symmetry is lowered. According to the symmetry of the model, we have to consider four, i.e. $(-\sqrt{3}/2, -\frac{1}{2})$, vertical ($y$), $(-\frac{1}{2}, -\sqrt{3}/2)$ and horizontal (x) decimation directions (see figure 1).

3. Results

In this section, the exact decimation procedure will be carried out for the anisotropic Kagome lattice GMS model in four different directions, which leads to four different elementary cells shown in figure 2.

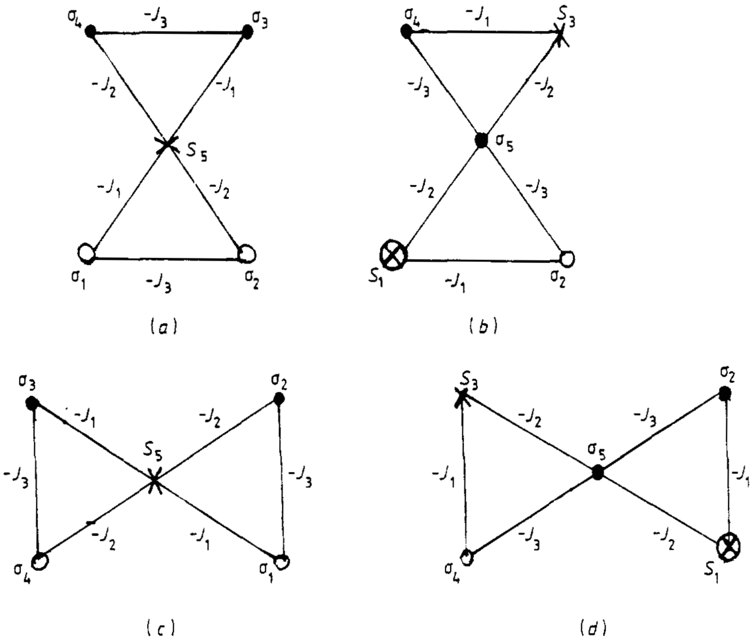

Figure 2. Elementary cells for the four different decimation directions: $(-\sqrt{3}/2, -\frac{1}{2})$ (a), vertical (b), $(-\frac{1}{2}, -\sqrt{3}/2)$ (c) and horizontal (d). The summations are taken over full circles (●) and crosses (×) only. Spin-$\frac{1}{2}$ and spin-1 Ising objects which are not to be summed over are indicated by open circles (○) and crosses surrounded by circles (⊗), respectively.

Following Jaekel and Maillard (1985), the corresponding criteria in the cases of figures 2(a), (b), (c) and (d), respectively, are

$$\sum_{\sigma'_3,\sigma'_4,S_5} W(\sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4, S_5) = F(K, L, M, \Delta) \tag{2}$$

$$\sum_{S_3,\sigma'_4,\sigma'_5} W'(S_1, \sigma'_2, S_3, \sigma'_4, \sigma'_5) = F(K, L, M, \Delta') \exp(\Delta^* S_1^2) \tag{3}$$

$$\sum_{\sigma'_2,\sigma'_3,S_5} W(\sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4, S_5) = F(K, L, M, \Delta) \tag{4}$$

$$\sum_{\sigma'_2,S_3,\sigma'_5} W'(S_1, \sigma'_2, S_3, \sigma'_4, \sigma'_5) = F(K, L, M, \Delta') \exp(\Delta^* S_1^2) \tag{5}$$

with Boltzmann weights

$$W = \exp[KS_5(\sigma'_1 + \sigma'_3) + LS_5(\sigma'_2 + \sigma'_4) + M(\sigma'_1\sigma'_2 + \sigma'_3\sigma'_4) + \Delta S_5^2] \tag{6}$$

$$W' = \exp[K(S_1\sigma'_2 + S_3\sigma'_4) + L\sigma'_5(S_1 + S_3) + M\sigma'_5(\sigma'_2 + \sigma'_4) + \Delta' S_3^2] \tag{7}$$

where $K = \frac{1}{2}J_1/kT$, $L = \frac{1}{2}J_2/kT$, $M = \frac{1}{4}J_3/kT$, $\Delta = G/kT$, $\Delta^* = G^*/kT$, $\Delta' = G'/kT$ and $\sigma'_i = 2\sigma_i = \pm 1$.

If particular boundary conditions, which do not modify the partition function per site at the disorder domain in the thermodynamic limit, are introduced on the first layer along the decimation direction and periodic boundary conditions are imposed in the other direction, the decimation procedures defined by (2)-(5) can be iterated until all the spins of the system have been summed over (cf Jaekel and Maillard 1985). This leads to the following results for the partition function per site:

$$f=(F)^{1/3}$$
$$=(4\mathrm{e}^{M}\{\cosh M+\mathrm{e}^{\Delta}\cosh(K+L)[\mathrm{e}^{M}\cosh(K+L)+\mathrm{e}^{-M}\cosh(K-L)]\})^{1/3} \qquad (8)$$

which is valid under the constraint

$$\mathrm{e}^{\Delta}=\frac{\sinh 2M}{\mathrm{e}^{-2M}\cosh^2(K-L)-\mathrm{e}^{2M}\cosh^2(K+L)} \qquad (9)$$

and confined to the regions

$$J_1J_2J_3<0 \qquad (10)$$

$$\tfrac{1}{2}|J_3|<|J_1|,\ |J_2| \qquad (11)$$

and

$$T\leqslant T_D \qquad (12)$$

where T_D is the temperature defined by

$$\mathrm{e}^{2M}=\frac{\cosh(K-L)}{\cosh(K+L)} \qquad (13)$$

and

$$f=(F)^{1/3}$$
$$=\{4\cosh^2 M+4\mathrm{e}^{\Delta}\cosh(K+L)[\mathrm{e}^{2M}\cosh(K+L)+\cosh(K-L)]-\sinh 2M)\}^{1/3} \qquad (14)$$

which is valid along the trajectory

$$\mathrm{e}^{2M}=\left|\frac{\sinh(K-L)}{\sinh(K+L)}\right| \qquad (15)$$

with any real value of G and confined to the region

$$J_1J_2J_3<0. \qquad (16)$$

4. Discussion

It follows from (9), (10), (11) and (13) that G becomes infinite when $T=T_D$, and accordingly the GMS model reduces to a pure spin-$\frac{1}{2}$ model. After eliminating an infinite factor $\exp(\Delta)$, we have

$$f=[4(\cosh 2K+\cosh 2L)]^{1/3} \qquad (17)$$

when

$$\mathrm{e}^{2M}=\frac{\cosh(K-L)}{\cosh(K+L)} \qquad (18)$$

which can be shown to be equivalent to Jaekel and Maillard's equation (7) with $q=2$ and

$$f=\left(\frac{4\sinh^2 2K}{\cosh 2K-\cosh 2L}\right)^{1/3} \tag{19}$$

when

$$e^{2M}=\left|\frac{\sinh(K-L)}{\sinh(K+L)}\right|. \tag{20}$$

The methods used so far for obtaining disorder solutions rely on the same mechanism: a certain local decoupling of the degrees of freedom of the model, which results in an effective reduction of dimensionality for the lattice system. In our problem, it can be expressed more concretely as

$$\mathscr{K}+K=0 \tag{21}$$

where $\mathscr{K}$ is the effective interaction obtained by summing the upper three spins in the elementary cell and K is the interaction between the other two spins. In the cases of figures 2(a) and (c), equation (21) leads to a unique condition on the parameters of the model and for figures 2(b) and (d), equation (21) can be satisfied only for infinite G, which leads to two conditions.

The conditions (9) and (15) can be satisfied simultaneously for a spin-$\frac{1}{2}$ and spin-1 mixed-spin model (i.e. $G=0$). In this case, it is easy to show that the intra-row correlation is zero in both horizontal and vertical directions. As pointed out by Maillard (1986), the disorder temperature corresponds to the temperature where even that short-range order vanishes.

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